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The Physics Superselection Principle in Vertex Operator Algebra Theory

Haisheng Li*

Department of Mathematical Sciences, Rutgers University, Camden, New Jersey 08102

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We formulate an interpretation of the theory of physics superselection sectors in terms of vertex operator algebra language and prove some initial results. As one of the main results we give a construction of simple currents from a weight-one primary semisimple element. By applying our results to vertex operator algebras associated to affine Lie algebras or to positive-definite even lattices, we construct their simple currents. © 1997 Academic Press

1. INTRODUCTION

In mathematical physics, there are various approaches to two-dimensional quantum field theory, among which are the $\mathbb{C}^*$-algebra approach through the superselection principle [HK] and the chiral algebra (vertex operator algebra) approach [BPZ, MSe]. This paper studies the application of the theory of superselection sectors to vertex operator algebra theory [B, FLM2, FHL]. Although this paper is motivated by some mathematical physics papers such as [FRS, MSc] in the theory of superselection sectors, the main results in this paper are purely algebraic in the context of vertex operator algebras.

In local quantum field theory one considers a Hilbert space H of physical states which decomposes into inequivalent, irreducible modules H_i (superselection sectors) for the observable algebra $\mathcal{A}$, possibly with some multiplicities [HK]. Among the superselection sectors, there is a distinguished sector H_0 which contains the vacuum vector and carries the vacuum representation π_0. In general, $\mathcal{A}$ admits infinitely many inequiva-

* E-mail address: hli@math.ias.edu.

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lent irreducible modules, so a criterion is needed to rule out the physically irrelevant modules.

If U is an element of A with a left inverse U^*, then we have an endomorphism ψ_U of A defined by $\psi_U(a) = UaU^*$ for any $a \in A$. Consequently, we have a representation $\pi_0 \psi_U$ on H_0 of A. Roughly speaking, in the algebraic theory of superselection sectors (see, e.g., [HK]), the physics superselection sectors by definition consist of each equivalent class of irreducible representation π which is equivalent to $\pi_0 \psi_U$ for some (so-called localized) endomorphism ψ_U of A. If $W_i = (H_0, \pi_0 \psi_i)$ $(i = 1, 2)$ are superselection sectors, an intertwiner from W_1 to W_2 is defined to be an A-homomorphism, and $W = (H_0, \pi_0 \psi_2 \psi_1)$ is defined to be the tensor product module of W_1 with W_2. Such a tensor product module is in general reducible, but it is assumed to be decomposable into irreducible ones. Furthermore, if $(H_0, \pi_0 \psi_i)$ $(i = 1, 2, 3)$ are three superselection sectors, then an intertwining operator of type $\binom{W_3}{W_1 W_2}$ is defined to be an intertwiner or an A-homomorphism ϕ from $W = (H_0, \pi_0 \psi_2 \psi_1)$ to W_3 and fusion rules are defined accordingly.

In mathematics, the notion of vertex operator algebra [B, FLM2] naturally arose from the vertex operator construction of the moonshine module [FLM1] for the Monster group, the largest sporadic finite simple group. On the other hand, vertex operator algebras are essentially chiral algebras formulated in [BPZ] in two-dimensional conformal field theory. Vertex operator algebras provide a powerful algebraic tool for studying the general structure of conformal field theory. For vertex operator algebra theory, the notions of module, intertwining operator, and fusion rule have been defined in [FLM2, FHL]. Furthermore, the notions of tensor product for modules have been also developed in [HL0−HL3, Hua, Li3]. The purpose of this paper is to apply the physics superselection theory to vertex operator algebra theory [B, FLM2, FHL].

Note that if σ is an endomorphism of a vertex operator algebra V, then by definition σ preserves both the vacuum and the Virasoro element so that σ preserves each homogeneous subspace of V (see [FHL, Sect. 2.4]). If V is simple, i.e., V is an irreducible V-module, it follows from the Schur lemma that any nonzero endomorphism is a scalar. Then the twisting of V by σ is isomorphic to V. Therefore, it is impossible to obtain all irreducible modules by twisting V unless V is holomorphic, i.e., any irreducible V-module is isomorphic to V. Having known the above fact, we turn to a certain associative algebra.

For any vertex operator algebra V, Frenkel and Zhu [FZ] constructed a topological $\mathbb{Z}$-graded associative algebra $U(V)$, which was called the universal enveloping algebra of V. Roughly speaking, $U(V)$ is the associative algebra with identity generated by all $a(n)$ (linear in a), for $a \in V$, $n \in \mathbb{Z}$

with certain defining relations coming from the Jacobi identity and the Virasoro algebra relations. Then there is a natural 1-1 correspondence between the set of equivalence classes of lower truncated $\mathbb{Z}$-graded weak V-modules and the set of equivalence classes of continuous[1] lower truncated $\mathbb{Z}$-graded $U(V)$-modules. It is reasonable to believe that $U(V)$ should play the role of the observable algebra in the algebraic quantum field theory.

Note that for all known rational vertex operator algebras (the definition is given in Section 2) there are only finitely many inequivalent irreducible modules and all irreducible modules are exactly those which are needed in conformal field theory. For instance, it was proved [DL, FZ, Li1] that for any positive integer l, the set of equivalence classes of irreducible $L(l,0)$-modules is exactly the set of equivalence classes of unitary highest weight $\tilde{\mathbf{g}}$-modules of level l. It was also proved [DMZ, W] that if $c = 1 - 6(p - q)^2/pq$, where $p, q \in \{2, 3, \dots\}$ and p and q are relatively prime, then the set of equivalence classes of irreducible $L(c,0)$-modules is exactly the set of equivalence classes of lowest weight Virasoro modules in the minimal series given in [BPZ]. Therefore, at least for a rational vertex operator algebra V, each irreducible V-module or sector is physically relevant so that each irreducible V-module should be a superselection sector. Based on this interpretation we conjecture that each irreducible V-module is isomorphic to some twisting of V by an endomorphism of $U(V)$.

As mentioned, twisting the adjoint module by an endomorphism of a vertex operator algebra V does not give a new module. Notice that an endomorphism of V is an element of $\mathrm{End}_{\mathbb{C}} V$ satisfying certain conditions. Now we consider certain elements $\Delta(z) \in (\mathrm{End}_{\mathbb{C}} V)\{z\}$ satisfying certain conditions so that $(V, Y(\Delta(z), \cdot\,, z))$ is a V-module. (To study the twisting of intertwining operators we more generally consider $\Delta(z) \in U(V)\{z\}$.) We prove that $(V, Y(\Delta(z)\,\cdot\,, z))$ is a weak V-module if and only if the conditions (2.17)–(2.20) hold. This implies that such a $\Delta(z)$ induces an endomorphism ψ of $U(V)$ defined by

$$\psi(Y(a,z)) = Y(\Delta(z)a, z) \qquad \text{for any } a \in V. \tag{1.1}$$

Our first theorem claims that if $\Delta(z)$ is invertible, $\tilde{M}$ is isomorphic to a tensor product module of M with $\tilde{V}$ in the sense of [HL0–HL3, Hua, Li3]. This implies that $\tilde{V}$ is a simple current [SY1, SY2, FG], i.e., the tensor functor associated to $\tilde{V}$ gives a permutation on the set of equivalence classes of irreducible V-modules.

In Section 3, we construct such a $\Delta(h,z)$ satisfying the conditions (2.17)–(2.20) from a primary weight-one semisimple element h of a vertex operator algebra V. Applying our results to a vertex operator algebra $L(l,0)$ associated to an affine Lie algebra $\tilde{\mathbf{g}}$, we prove that if the funda-

[1] It was pointed out by C. Dong that this condition is necessary.

mental (dominant integral) weight λ_i is cominimal [FG], then for any complex number $l \neq -\Omega$ (the dual Coxeter number), $L(l, l\lambda_i)$ is a (weak) $L(l, 0)$-module and it is a simple current if l is a positive integer. This result has been proved in [FG] by calculating four-point functions (see also [F, FGV1, FGV2]). We also apply our results to the vertex operator algebra V_L associated to a positive-definite even lattice L to find all the fusion rules. This result has been previously obtained in [DL] by using a different method.

2. THE SUPERSELECTION PRINCIPLE IN TERMS OF VERTEX OPERATOR ALGEBRAS AND MODULES

In this section we formulate an interpretation of the physics superselection principle in terms of vertex operator algebras and modules and prove some initial results.

We recall the following definition from [FLM2, Sect. 8.10]. A *vertex operator algebra* is a $\mathbb{Z}$-graded vector space

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}; \qquad \text{for } v \in V_{(n)}, n = \operatorname{wt} v; \tag{2.1}$$

such that

$$\dim V_{(n)} < \infty \qquad \text{for } n \in \mathbb{Z}, \tag{2.2}$$

$$V_{(n)} = 0 \qquad \text{for } n \text{ sufficiently small}, \tag{2.3}$$

equipped with a linear map

$$V \to (\operatorname{End} V)[[z, z^{-1}]]$$

$$v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \qquad (\text{where } v_n \in \operatorname{End} V) \tag{2.4}$$

and with two distinguished homogeneous vectors $\mathbf{1}, \omega \in V$, satisfying the following conditions for $u, v \in V$,

$$u_n v = 0 \qquad \text{for } n \text{ sufficiently large}; \tag{2.5}$$

$$Y(\mathbf{1}, z) = 1; \tag{2.6}$$

$$Y(v, z)\mathbf{1} \in V[[z]] \qquad \text{and} \qquad \lim_{z \to 0} Y(v, z)\mathbf{1} = v; \tag{2.7}$$

$$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) & Y(u, z_1)Y(v, z_2) \\ & - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y(v, z_2)Y(u, z_1) \\ & = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y(Y(u, z_0)v, z_2); \end{aligned} \tag{2.8}$$

(the Jacobi identity) where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and where $\delta((z_1 - z_2)/z_0)$ is to be expanded as a formal power series in the second term in the numerator, z_2, and analogously for the other δ-function expressions; when each expression in (2.8) is applied to any element of V, the coefficient of each monomial in the formal variables is a finite sum;

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V) \tag{2.9}$$

for $m, n \in \mathbb{Z}$, where

$$L(n) = \omega_{n+1} \qquad \text{for } n \in \mathbb{Z}, \text{ i.e., } Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2} \tag{2.10}$$

and

$$\operatorname{rank} V \in \mathbb{C}; \tag{2.11}$$

$$L(0)v = nv = (\operatorname{wt} v)v \qquad \text{for } n \in \mathbb{Z} \text{ and } v \in V_{(n)}; \tag{2.12}$$

$$\frac{d}{dz} Y(u, z) = Y(L(-1)u, z). \tag{2.13}$$

This completes the definition. We shall just use V for a vertex operator algebra.

A *weak* V-module is a vector space M together with a linear map $Y_M(\cdot, z)$ from V to $(\operatorname{End} M)[[z, z^{-1}]]$ such that $Y_M(\mathbf{1}, z) = 1, Y_M(L(-1)a, z) = (d/dz)Y_M(a, z)$ for any $a \in V$ and that a suitably adjusted Jacobi identity holds. We shall just use M for the weak module. A subspace U of M is called a submodule if $Y_M(a, z)u \in U((z))$ for any $a \in V$, $u \in U$. If $\{0\}$ and M are the only submodules, M is said to be *irreducible*. Let $(W_i, Y_{W_i}(\cdot, z))$ $(i = 1, 2)$ be two weak V-modules. A *V-homomorphism* from W_1 to W_2 is a linear map ψ such that $\psi(Y_{W_1}(a, z)u) = Y_{W_2}(a, z)\psi(u)$ for any $a \in V$, $u \in W_1$. Furthermore, if ψ is a linear isomorphism, ψ is called a *V-isomorphism*.

A *V-module* is a weak V-module M which is a $\mathbb{C}$-graded vector space $M = \coprod_{h \in \mathbb{C}} M_{(h)}$, where $M_{(h)}$ is the eigenspace of $L(0)$ on M with eigenvalue h, such that for any $h \in \mathbb{C}$, $\dim M_{(h)} < \infty$ and $M_{(n+h)} = 0$ for sufficiently large integer n.

A *lower truncated $\mathbb{Z}$-graded weak V-module* is a weak V-module M together with a $\mathbb{Z}$-grading $M = \coprod_{n \in \mathbb{Z}} M(n)$ such that $M(n) = 0$ for sufficiently small integer n and that

$$a_n M(m) \subseteq M(m + k - n - 1) \qquad \text{for } a \in V_{(k)}, m, n, k \in \mathbb{Z}.$$

If any lower truncated $\mathbb{Z}$-graded weak V-module is a direct sum of irreducible $\mathbb{Z}$-graded weak V-modules, we say that V is *rational*. If any weak V-module W is a direct sum of irreducible (ordinary) V-modules, we say that V is *regular*. It was proved in [DLM] that vertex operator algebras V_L associated to a nondegenerate even lattice L, $L(l,0)$ associated to standard modules of a positive integral level l for an affine Lie algebra $\tilde{\mathbf{g}}$, $L(c,0)$ associated to unitary highest weight Virasoro modules with central charge $0 < c < 1$, and $V^\natural$, Frenkel, Lepowsky, and Meurman's moonshine module, are regular. That is, all known rational vertex operator algebras are regular.

Let W_i $(i = 1, 2, 3)$ be three weak V-modules. Then *an interesting operator* of type $\binom{W_3}{W_1 W_2}$ is defined in [FHL] to be a linear map $I(\cdot, z)$ from W_1 to $\mathrm{Hom}_{\mathbb{C}}(W_2, W_3)\{z\}$ such that $I(L(-1)u, z) = (d/dz)I(u, z)$ for $u \in W_1$ and that a suitably adjusted Jacobi identity holds.

Denote by $I\binom{W_3}{W_1 W_2}$ the space of all intertwining operators of the indicated type. The dimension of this vector space is called the *fusion rule* of this type. Let $\mathrm{Irr}(V)$ be the set of equivalence classes of irreducible weak V-modules and for any V-module M, denote the equivalence class of M by $E(M)$.

Let V be a vertex operator algebra. Recall Frenkel and Zhu's construction of the universal enveloping algebra $U(V)$ of V as follows [FZ, Sect. 1.3]: First, let A be the free algebra with generators $a(n)$ (linear in a) for $a \in V$, $n \in \mathbb{Z}$. Define $\deg(a(n)) = m - n - 1$ for $a \in V_{(m)}$, $n \in \mathbb{Z}$. Then A becomes a $\mathbb{Z}$-graded algebra with $A = \coprod_{n \in \mathbb{Z}} A_n$. For any $n \in \mathbb{Z}$, $i \in \mathbb{Z}$, we set $I_{n,i} = \sum_{j \geq i} A_{n+j} A_{-j}$ $(\subseteq A_n)$. Then

$$\cdots I_{n,2} \subseteq I_{n,1} \subseteq I_{n,0} \subseteq I_{n,-1} \subseteq I_{n,-2} \subseteq \cdots.$$

Let $i, j \in \mathbb{Z}$, $u, v \in A_n$ for a fixed n. Then for any $x \in (u + I_{n,i}) \cap (v + I_{n,j})$, we have

$$x + I_{n,k} \subseteq (u + I_{n,i}) \cap (v + I_{n,j}) \qquad \text{for } k \geq i, j.$$

Thus $(u + I_{n,i}) \cap (v + I_{n,j}) = \bigcup_x (x + I_{n,k})$, where x runs through each element of $(u + I_{n,i}) \cap (v + I_{n,j})$. Let τ be the collection of the empty set and all unions of some $u + I_{n,i}$ for $u \in A_n$, $i \in \mathbb{Z}$. Then τ is closed under finite intersection, so that τ defines a topology on A_n. Thus a sequence $\{x_m\}$ of elements in A_n converges to zero if and only if for any $I_{n,k}$ there exists a positive integer r such that $x_m \in I_{n,k}$ for $m \geq r$. Since

$$A_m I_{n,i} \subseteq I_{m+n,i}, \qquad I_{n,i} A_m \subseteq I_{m+n,i-m}$$

for $m, n \in \mathbb{Z}$, the multiplication of A is continuous.

Let $\bar{A}_n$ be the completion of A_n under this topology. Set $\bar{A} = \coprod_{n\in\mathbb{Z}} \bar{A}_n$. Then $\bar{A}$ is a topological $\mathbb{Z}$-graded associative algebra. Next, we define $U(V)$ to be the quotient algebra of $\bar{A}$ modulo the two-sided ideal generated by the relations

$$\mathbf{1}(n) = \delta_{n,-1}; \tag{2.14}$$

$$(L(-1)a)(n) = -na(n-1); \tag{2.15}$$

$$\begin{aligned}\sum_{i=0}^{\infty} (-1)^i \binom{m}{i} a(m-n-1-i)b(i-k-1) \\ - \sum_{i=0}^{\infty} (-1)^{m-i} \binom{m}{i} b(m-k-i-1)a(i-n-1) \\ = \sum_{i=0}^{\infty} \binom{-n-1}{i} (a_{i-m-1}b)(-n-k-2-i)\end{aligned} \tag{2.16}$$

for $a, b \in V, m, n, k \in \mathbb{Z}$. Then $U(V) = \coprod_{n\in\mathbb{Z}} U(V)_n$ is a $\mathbb{Z}$-graded topological associative algebra such that there is a natural 1-1 correspondence between the set of equivalence classes of the lower truncated $\mathbb{Z}$-graded continuous $U(V)$-module and the set of equivalence classes of the lower truncated $\mathbb{Z}$-graded weak V-module.

PROPOSITION 2.1. *Let V be a vertex operator algebra and let $\Delta(z) \in (\mathrm{End}_{\mathbb{C}} V)[[z, z^{-1}]]$ such that*

$$\Delta(z)a \in V[z, z^{-1}] \qquad \textit{for any } a \in V. \tag{2.17}$$

Suppose that

$$\left(\tilde{V}, \tilde{Y}(\cdot, z)\right) := (V, Y(\Delta(z)\cdot, z))$$

is a weak V-module. Then the following conditions hold:

$$\Delta(z)\mathbf{1} = \mathbf{1}; \tag{2.18}$$

$$[L(-1), \Delta(z)] = -\frac{d}{dz}\Delta(z); \tag{2.19}$$

$$Y(\Delta(z_2+z_0)a, z_0)\Delta(z_2) = \Delta(z_2)Y(a, z_0) \qquad \textit{for any } a \in V. \tag{2.20}$$

Conversely, if the above three conditions hold, for any weak V-module $(M, Y_M(\cdot, z))$

$$\left(\tilde{W}, Y_{\overline{M}}(\cdot, z)\right) := (W, Y_M(\Delta(z)\cdot, z))$$

is a weak V-module.

Proof. First of all, because of (2.17),

$$Y(\Delta(z)a, z)b \in V((z)) \qquad \text{for any } a, b \in V.$$

We prove the converse first. Suppose that (2.18)–(2.20) hold. Let (W, Y_W) be a weak V-module. Then for any $a, b \in V$ we have

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y_W(\Delta(z_1)a, z_1)Y_W(\Delta(z_2)b, z_2) \\ & \quad - z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)Y_W(\Delta(z_2)b, z_2)Y_W(\Delta(z_1)a, z_1) \\ & \quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_W\big(Y(\Delta(z_1)a, z_0)\Delta(z_2)b, z_2\big) \\ & \quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_W\big(Y(\Delta(z_2 + z_0)a, z_0)\Delta(z_2)b, z_2\big) \\ & \quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y_W\big(\Delta(z_2)Y(a, z_0)b, z_2\big). \end{aligned} \tag{2.21}$$

This proves the Jacobi identity. Furthermore, the conditions (2.18) and (2.19) imply

$$\begin{aligned} Y_W(\Delta(z)\mathbf{1}, z) &= 1, \\ [L(-1), Y_W(\Delta(z)a, z)] &= \frac{d}{dz}Y_W(\Delta(z)a, z) \text{ for } a \in V. \end{aligned} \tag{2.22}$$

Therefore $(\tilde{W}, Y_{\tilde{W}}(\cdot, z))$ is a weak V-module.

On the other hand, suppose that $(V, Y(\Delta(z)\cdot, z))$ is a weak V-module. Since $\tilde{Y}(\mathbf{1}, z) = 1$, using the skew-symmetry we get

$$e^{zL(-1)}\Delta(z)\mathbf{1} = Y(\Delta(z)\mathbf{1}, z)\mathbf{1} = \mathbf{1}.$$

Thus $\Delta(z)\mathbf{1} = \mathbf{1}$. Since $\tilde{Y}(L(-1)a, z) = (d/dz)\tilde{Y}(a, z)$ for $a \in V$ we have

$$\begin{aligned} Y(\Delta(z)L(-1)a, z) &= Y(L(-1)a, z) \\ &= Y(L(-1)\Delta(z)a, z) + Y(\Delta'(z)a, z). \end{aligned}$$

That is,

$$Y([L(-1), \Delta(z)]a, z) = -Y(\Delta'(z)a, z).$$

By a similar argument we obtain $[L(-1), \Delta(z)] = -\Delta'(z)$. Following the argument of (2.21) we find that the Jacobi identity for $\tilde{Y}$ holds if and only if

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y\big(Y(\Delta(z_2 + z_0)a, z_0)\Delta(z_2)b, z_2\big)$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y\big(\Delta(z_2)Y(a, z_0)b, z_2\big). \tag{2.23}$$

Applying (2.23) to **1**, then using the skew-symmetry we obtain

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y\big(\Delta(z_1)a, z_0\big)\Delta(z_2) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\Delta(z_2)Y(a, z_0). \tag{2.24}$$

Then we immediately obtain (2.20). This proves that the listed conditions are also necessary and concludes the proof. ∎

It follows from Proposition 2.1 that each $\Delta(z)$ gives rise to an endomorphism ψ of $U(V)$ such that $\psi(Y(a, z)) = Y(\Delta(z)a, z)$ for any $a \in V$.

The following proposition gives the injective property of $\Delta(z)$ as a linear map from V to $V \otimes \mathbb{C}((z))$.

PROPOSITION 2.2. *Let V be a simple vertex operator algebra and let $\Delta(z) \in (\mathrm{End}_{\mathbb{C}} V)[[z, z^{-1}]]$ satisfy the conditions* (2.17)–(2.20). *Then for $a \in V$, $\Delta(z)a = 0$ if and only if $a = 0$.*

Proof. Set

$$\ker \Delta(z) := \{a \in V \mid \Delta(z)a = 0\}.$$

Then it is equivalent to proving that $\ker \Delta(z) = 0$. Let $a \in \ker \Delta(z)$. Then for any $u \in V$, using (2.20) we get

$$\Delta(z_2)Y(u, z_0)a = Y\big(\Delta(z_2 + z_0)u, z_0\big)\Delta(z_2)a = 0.$$

Thus $\ker \Delta(z)$ is an ideal of V. Since V is simple and $\Delta(z)\mathbf{1} = \mathbf{1}$, we obtain $\ker \Delta(z) = 0$. This concludes the proof. ∎

Notice that for any (weak) V-module W, $Y_W(\cdot, z)$ is an intertwining operator of type $\binom{W}{VW}$. Then Proposition 2.1 implies that $\tilde{Y}_W(\cdot, z)$ is an intertwining operator of type $\binom{\tilde{W}}{V\tilde{W}}$. To generalize this for an intertwining operator for three arbitrary modules we shall consider elements $\Delta(z) \in U(V)\{z\}$ so that $\Delta(z)$ can act on any weak V-module. This leads us to the following definition:

DEFINITION 2.3. Let V be a vertex operator algebra. Then we define $G(V)$ to be the set consisting of each $\Delta(z)=\sum_{r\in\mathbb{C}}\Delta_r z^r\in U(V)\{z\}$ satisfying the following condition for any weak V-module W and any $u\in W$: There exist (finitely many) $n_1,\dots,n_k\in\mathbb{C}$ such that

$$\Delta(z)u\in z^{n_1}W[z]+\cdots+z^{n_k}W[z] \tag{2.25}$$

and all the conditions in Proposition 2.1 hold.

The following propositions are motivated by the theory of superselection sectors:

PROPOSITION 2.4. *Let $\Delta(z)\in G(V)$, let M^i $(i=1,2,3)$ be three weak V-modules, and let $I(\cdot,z)$ be an intertwining operator of type $\binom{M^3}{M^1M^2}$. Then $\tilde{I}(\cdot,z)=I(\Delta(z)\cdot,z)$ is an intertwining operator of type $\binom{\tilde{M}^3}{M^1\tilde{M}^2}$.*

Proof. The $L(-1)$-derivative property for $\tilde{I}(\cdot,z)$ follows from the condition (2.19) immediately. For any $a\in V$, $u\in M^1$ we have

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{\tilde{M}^3}(a,z_1)\tilde{I}(u,z_2)-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)\tilde{I}(u,z_2)Y_{\tilde{M}^2}(a,z_1)\\
&\quad=z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y_{M^3}(\Delta(z_1)a,z_1)I(\Delta(z_2)u,z_2)\\
&\qquad-z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)I(\Delta(z_2)u,z_2)Y_{M^2}(\Delta(z_1)a,z_1)\\
&\quad=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I(Y_{M^1}(\Delta(z_1)a,z_0)\Delta(z_2)u,z_2)\\
&\quad=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I(\Delta(z_2)Y_{M^1}(a,z_0)u,z_2)\\
&\quad=z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\tilde{I}(Y_{M^1}(a,z_0)u,z_2).
\end{aligned} \tag{2.26}$$

Then the proof is complete. ∎

PROPOSITION 2.5. *Let $\Delta(z)\in G(V)$ and let ψ be a V-homomorphism from a (weak) V-module W to another V-module M. Then ψ is also a V-homomorphism from the V-module $\tilde{W}$ to the V-module $\tilde{M}$.*

Proof. For any $a \in V$, $u \in W$, we have

$$\begin{aligned}\psi\left(Y_{\tilde{W}}(a,z)u\right) &= \psi\left(Y_W(\Delta(z)a,z)u\right)\\ &= Y_M(\Delta(z)a,z)\psi(u)\\ &= Y_{\tilde{M}}(a,z)\psi(u).\end{aligned} \tag{2.27}$$

Thus ψ is a V-homomorphism from $\tilde{W}$ to $\tilde{M}$. ∎

PROPOSITION 2.6. *Let $\Delta_1(z), \Delta_2(z) \in G(V)$. Then $\Delta_1(z)\Delta_2(z) \in G(V)$.*

Proof. By definition we have

$$\Delta_1(z)\Delta_2(z)\mathbf{1} = \Delta_1(z)\mathbf{1} = \mathbf{1}, \tag{2.28}$$

$$\begin{aligned}&[L(-1), \Delta_1(z)\Delta_2(z)]\\ &\quad = [L(-1), \Delta_1(z)]\Delta_2(z) + \Delta_1(z)[L(-1), \Delta_2(z)]\\ &\quad = -\left(\frac{d}{dz}\Delta_1(z)\right)\Delta_2(z) - \Delta_1(z)\left(\frac{d}{dz}\Delta_2(z)\right)\\ &\quad = -\frac{d}{dz}(\Delta_1(z)\Delta_2(z)),\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}&z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y(\Delta_1(z_1)\Delta_2(z_1)a, z_0)\Delta_1(z_2)\Delta_2(z_2)\\ &\qquad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\Delta_1(z_2)Y(\Delta_2(z_1)a, z_0)\Delta_2(z_2)\\ &\qquad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\Delta_1(z_2)\Delta_2(z_2)Y(a, z_0)\end{aligned} \tag{2.30}$$

for any $a \in V$. The other conditions follow directly from the definition. Thus $\Delta_1(z)\Delta_2(z) \in G(V)$. ∎

It is clear that $id_V \in G(V)$, so that $G(V)$ is a semigroup.

PROPOSITION 2.7. *Let $\Delta(z) \in G(V)$ such that $\Delta(z)$ has an inverse $\Delta^{-1}(z) \in U(V)\{z\}$. Then $\Delta^{-1}(z) \in G(V)$.*

Proof. First, we have $\Delta^{-1}(z)\mathbf{1} = \Delta^{-1}(z)\Delta(z)\mathbf{1} = \mathbf{1}$. Since $\Delta(z)\Delta^{-1}(z) = 1$, we have

$$0 = \left(\frac{d}{dz}\Delta(z)\right)\Delta^{-1}(z) + \Delta(z)\frac{d}{dz}\Delta^{-1}(z). \tag{2.31}$$

Then

$$\begin{aligned}\frac{d}{dz}\Delta^{-1}(z) &= -\Delta^{-1}(z)\left(\frac{d}{dz}\Delta(z)\right)\Delta^{-1}(z)\\ &= -L(-1)\Delta^{-1}(z) + \Delta^{-1}(z)L(-1)\\ &= -\left[L(-1), \Delta^{-1}(z)\right].\end{aligned} \tag{2.32}$$

For any $a \in V$, we have

$$\begin{aligned}&z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\Delta^{-1}(z_2)Y(a, z_0)\\ &\quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)\Delta^{-1}(z_2)Y\left(\Delta(z_1)\Delta^{-1}(z_1)a, z_0\right)\\ &\quad = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y\left(\Delta^{-1}(z_1)a, z_0\right)\Delta^{-1}(z_2).\end{aligned} \tag{2.33}$$

Thus $\Delta^{-1}(z) \in G(V)$. ∎

Define $G_l^0(V)$ to be the subset of $G(V)$, consisting of each $\Delta(z)$ that has a left inverse $\Delta^*(z)$ in $G(V)$. Denote by $G^0(V)$ the subgroup of all invertible elements of $G(V)$.

Conjecture 2.8. Let V be a vertex operator algebra. Then for any irreducible V-module $(M, Y_M(\cdot, z))$, there is a $\Delta(z) \in G_l^0(V)$ such that the V-module $(V, Y(\Delta(z)\cdot, z))$ is isomorphic to $(M, Y_M(\cdot, z))$.

As mentioned in the Introduction, this conjecture comes from our interpretation of the physics superselection principle in the context of vertex operator algebras. In Section 3 we shall prove that Conjecture 2.8 holds for vertex operator algebras V_L associated to positive-definite even lattices L.

In [HL0–HL3, Hua], a theory of a tensor product for the module category of a vertex operator algebra was developed and it involved geometry in a certain natural way. Later, a formal variable construction of tensor products was given in [Li3]. These two constructions give isomorphic tensor product modules although they appear differently. For our purpose, we recall the following definition of a tensor product for modules for a vertex operator algebra V from [Li3] (see [HL0–HL3] for a different version).

DEFINITION 2.9. Let M^1 and M^2 be two weak V-modules. A *tensor product* for the ordered pair (M^1, M^2) is a pair $(M, F(\cdot, z))$ consisting of a weak V-module M and an intertwining operator $F(\cdot, z)$ of type $\binom{M}{M^1 M^2}$ such that the following universal property holds: For any weak V-module W and any intertwining operator $I(\cdot, z)$ of type $\binom{W}{M^1 M^2}$, there exists a unique V-homomorphism ψ from M to W such that $I(\cdot, z) = \psi \circ F(\cdot, z)$. (Here ψ extends canonically to a linear map from $M\{z\}$ to $W\{z\}$.)

As a direct consequence of Definition 2.9 we have (see [Li3] for a proof):

COROLLARY 2.10. *If $(M, F(\cdot, z))$ is a tensor product for the ordered pair (M^1, M^2) of weak V-modules, then for any weak V-module M^3, $\mathrm{Hom}_V(M, M^3)$ is linearly isomorphic to the space of intertwining operators of type $\binom{M^3}{M^1 M^2}$.*

Let M be a V-module and let $\Delta(z) \in G_l^0(V)$. Set $W = V$, $Y_W(\cdot, z) = Y_V(\Delta(z)\cdot, z)$. In the theory of superselection sectors (see, for example, [MSc, Sect. 1]), essentially $(M, Y_M(\Delta(z)\cdot, z))$ was defined to be the tensor product module of M with W. Here we have

PROPOSITION 2.11. *Let $(W, F(\cdot, z))$ be a tensor product for a pair (M^1, M^2) of weak V-modules and let $\Delta(z) \in G^0(V)$. Then $(\tilde{W}, \tilde{F}(\cdot, z))$ is a tensor product of the pair $(M^1, \tilde{M}^2)$.*

Proof. From Proposition 2.4 we have an intertwining operator $\tilde{F}(\cdot, z) = F(\Delta(z)\cdot, z)$ of type $\binom{\tilde{W}}{M^1 \tilde{M}^2}$. Let M be any V-module and let $I(\cdot, z)$ be any intertwining operator of type $\binom{M}{M^1 \tilde{M}^2}$. Then $I(\Delta^{-1}(z)\cdot, z)$ is an intertwining operator of type $\binom{\hat{M}}{M^1 M^2}$, where $(\hat{M}, Y_{\hat{M}}(\cdot, z)) = (M, Y_M(\Delta^{-1}(z)\cdot, z))$. By the universal property of $(W, F(\cdot, z))$, there is a unique V-homomorphism ψ from $\tilde{W}$ to $\hat{M}$ such that $\hat{I}(\cdot, z) = \psi \circ \hat{F}(\cdot, z)$. By Proposition 2.5, ψ is a V-homomorphism from W to M. Since $\Delta(z)u$ only involves finitely many terms, we have

$$I(u, z) = I\big(\Delta(z)\Delta^{-1}(z)u, z\big) = \psi \circ F\big(\Delta(z)\Delta^{-1}(z)u, z\big) = \psi \circ F(u, z).$$

It is not difficult to check that ψ is the unique V-homomorphism from $\tilde{W}$ to $\hat{M}$ such that $\hat{I}(\cdot, z) = \psi \circ \hat{F}(\cdot, z)$. Then the proof is complete. ∎

COROLLARY 2.12. *Let M be a weak V-module and let $\Delta(z) \in G^0(V)$. Then the pair $(\tilde{M}, \tilde{I}(\cdot, z))$ is a tensor product of M with $\tilde{V}$, where $I(\cdot, z)$ is the* *intertwining operator of type* $\binom{M}{MV}$, *defined by* $I(u,z)a = e^{zL(-1)}Y_M(a,-z)u$ *for* $a \in V$, $u \in M$.

Proof. By Proposition 5.1.6 in [Li3], $(M, I(\cdot,z))$ is a tensor product for the pair (M, V). By Proposition 2.11, $(\tilde{M}, \tilde{I}(\cdot,z))$ is a tensor product for $(M, \tilde{V})$. ∎

In general, we have

Conjecture 2.13. Let M be a weak V-module and let $\Delta(z) \in G_l^0(V)$. Then the pair $(\tilde{M}, \tilde{I}(\cdot,z)) := (M, I(\Delta(z)\cdot, z))$ is a tensor product of $(M, \tilde{V})$, where $I(\cdot,z)$ is defined by $I(u,z)a = e^{zL(-1)}Y_M(a,-z)u$ for $a \in V$, $u \in M$.

The following definition is the algebraic counterpart of the physics definition of simple currents (see, for example, [SY1, SY2, FG]).

DEFINITION 2.14. Let V be a vertex operator algebra. An irreducible weak V-module M is called a *simple current* if for any irreducible weak V-module W, there exists a tensor product of M and W, which is irreducible.

If the associativity of the tensor product is assumed, then one can show that M being a simple current is equivalent to that the tensor functor "$M\times$" is a permutation acting on the set of equivalence classes of irreducible weak V-modules. By Corollary 2.12 we have:

THEOREM 2.15. *For any* $\Delta(z) \in G^0(V)$, $(V, Y(\Delta(z)\cdot, z))$ *is a simple current* V*-module*.

Proof. By definition, we only need to prove that $(\tilde{W}, Y_{\tilde{W}}(\cdot,z))$ is irreducible for any irreducible weak V-module $(W, Y_W(\cdot,z))$. If U is a submodule of $\tilde{W}$, then

$$Y_W(a,z)u = Y_{\tilde{W}}\big(\Delta^{-1}(z)a, z\big)u \in U((z)) \qquad \text{for } a \in V, u \in U.$$

Thus U is also a submodule of W. Since W is irreducible, $\tilde{W}$ is irreducible. ∎

In the last section we will apply this result to vertex operator algebras associated to affine Lie algebras later.

Next, we define $H(V)$ to be the subset of $G_l^0(V)$ consisting of each $\Delta(z)$ such that $(V, Y(\Delta(z)\cdot, z))$ is isomorphic to $(V, Y(\cdot,z))$.

PROPOSITION 2.16. *Let* $\Delta(z) \in G^0(V) \cap H(V)$ *and let* $(W, Y_W(\cdot,z))$ *be any irreducible weak* V*-module. Then* $(\tilde{W}, Y_{\tilde{W}}(\cdot,z)) := (W, Y(\Delta(z)\cdot, z))$ *is* V*-isomorphic to* W.

Proof. First, from [FHL, Sect. 5.4] we have an intertwining operator $I(\cdot,z)$ of type $\binom{W}{WV}$, defined by $I(u,z)a = e^{zL(-1)}Y_W(a,-z)u$ for any $a \in V$, $u \in W$. By Proposition 2.4, we obtain an intertwining operator $\tilde{I}(\cdot,z)$ of type $\binom{\tilde{W}}{W\tilde{V}}$. Let ψ be a V-isomorphism from V onto $\tilde{V}$. Then we obtain an intertwining operator $I_1(\cdot,z) := \tilde{I}(\cdot,z)\psi$ of type $\binom{\tilde{W}}{WV}$. Furthermore, we obtain an intertwining operator $I_2(\cdot,z)$ of type $\binom{\tilde{W}}{VW}$, defined by $I_2(u,z)v = e^{zL(-1)}I_1(v,-z)u$. Since $(d/dz)I_2(\mathbf{1},z) = I_2(L(-1)\mathbf{1},z) = 0$, $I_2(\mathbf{1},z)$ is a constant independent of z. Let $\phi := I_2(\mathbf{1},z)$. For any $a \in V$, $u \in W$ we have the Jacobi identity

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)Y(a,z_1)I_2(\mathbf{1},z_2)u - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right)I_2(\mathbf{1},z_2)Y(a,z_1)u$$
$$= z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I_2\big(Y(a,z_0)\mathbf{1},z_2\big)u.$$

Taking Res_{z_0} we obtain

$$Y(a,z_1)I_2(\mathbf{1},z_2)u - I_2(\mathbf{1},z_2)Y(a,z_1)u = 0.$$

That is, ϕ is a V-homomorphism from W to $\tilde{W}$. Furthermore, since

$$z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) - z_0^{-1}\delta\left(\frac{z_2-z_1}{-z_0}\right) = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)$$

and

$$Y(a,z_1)I_2(\mathbf{1},z_2)u = I_2(\mathbf{1},z_2)Y(a,z_1)u,$$

we obtain

$$z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)Y(a,z_1)I_2(\mathbf{1},z_2)u = z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)I_2\big(Y(a,z_0)\mathbf{1},z_2\big)u.$$

Taking $\mathrm{Res}_{z_0}\,\mathrm{Res}_{z_1} z_0^{-1}z_2^{-1}$ we obtain

$$I_2(a,z_2)u = Y(a,z_2+z_0)I_2(\mathbf{1},z_2)u = I_2(\mathbf{1},z_2)Y(a,z_2+z_0)u.$$

Since $I_2(\cdot,z) \neq 0$, it follows that $\phi = I_2(\mathbf{1},z) \neq 0$. Because both W and $\tilde{W}$ are irreducible (from the proof of Theorem 2.15), ϕ is an isomorphism. Then the proof is complete. ∎

COROLLARY 2.17. *Let $\Delta_1(z), \Delta_2(z) \in G^0(V)$ such that $(V, Y(\Delta_1(z)\cdot\,,z))$ is V-isomorphic to $(V, Y(\Delta_2(z)\cdot\,,z))$. Then for any irreducible weak V-module $(W, Y_W(\cdot,z))$, we have*

$$\big(W, Y_W(\Delta_1(z)\cdot\,,z)\big) \simeq \big(W, Y_W(\Delta_2(z)\cdot\,,z)\big).$$

Proof. Since $(V, Y(\Delta_1(z)\cdot, z)) \simeq (V, Y(\Delta_2(z)\cdot, z))$, we obtain

$$\left(V, Y\left(\Delta_1^{-1}(z)\Delta_2(z)\cdot, z\right)\right) \simeq (V, Y(\cdot, z)).$$

By Proposition 2.16 we have

$$\left(W, Y_W\left(\Delta_1^{-1}(z)\Delta_2(z)\cdot, z\right)\right) \simeq \left(W, Y_W(\cdot, z)\right).$$

Then

$$\left(W, Y_W\left(\Delta_1(z)\cdot, z\right)\right) \simeq \left(W, Y_W\left(\Delta_2(z)\cdot, z\right)\right). \quad \blacksquare$$

Remark 2.18. Clearly we have an equivalence relation $\equiv$ on $G_l^0(V)$ defined by $\Delta_1(z) \equiv \Delta_2(z)$ if and only if $(W, Y_W(\Delta_1(z)\cdot, z)) \simeq (W, Y_W(\Delta_2(z)\cdot, z))$ for any irreducible weak V-module $(W, Y_W(\cdot, z))$. We conjecture that Corollary 2.17 holds if $\Delta_1(z)$, $\Delta_2(z) \in G_l^0(V)$ such that $(V, Y(\Delta_1\cdot, z)) \simeq (V, Y(\Delta_2\cdot, z))$. If this is true, then combining with Conjecture 2.8 we would have a 1-1 correspondence between $\mathrm{Irr}(V)$ and $G_l^0(V)/\equiv$.

3. SIMPLE CURRENT MODULES

In this section, we first give a construction of $\Delta(z)$ from a primary weight-one semisimple element of a vertex operator algebra. Then we apply this result to vertex operator algebras associated to a positive-definite even lattice or to affine Lie algebras.

Recall the following commutator formula for a vertex operator algebra V ([FLM2, 8.6.5]; see also [B]),

$$[a_m, b_n] = \sum_{i=0}^{\infty} \binom{m}{i} (a_i b)_{m+n-i} \tag{3.1}$$

for $a, b \in V$, $m, n \in \mathbb{Z}$.

PROPOSITION 3.1. *Let V be a regular vertex operator algebra and let $a \in V$ such that $L(n)a = \delta_{n,0}a$ for $n \in \mathbb{Z}_+$ and that a_0 acts semisimply on V. Then a_0 acts semisimply on any weak V-module.*

Proof. Since V is regular (any weak V-module is a direct sum of irreducible V-modules with finite-dimensional homogeneous subspaces), it is enough to prove that a_0 acts semisimply on each irreducible V-module W. From the commutator formula (3.1) we get $[L(0), a_0] = 0$, so that a_0 preserves each homogeneous subspace of W. Since W has finite-dimensional homogeneous subspaces, there exists a $0 \neq u \in W$ such that $a_0 u = r\lambda$ for some $\lambda \in \mathbb{C}$. Since W is irreducible, W as a V-module is generated by u. If $a_0 b = rb$ for $b \in V$, $r \in \mathbb{C}$, then $[a_0, b_n] = (a_0 b)_n = rb_n$ for $n \in \mathbb{Z}$. Then it follows that a_0 acts semisimply on W. Then the proof is complete. $\blacksquare$

Let V be a regular vertex operator algebra and let $h \in V$ satisfy the conditions

$$L(n)h = \delta_{n,0}h, \qquad h_n h = \delta_{n,1}\gamma \mathbf{1} \text{ for any } n \in \mathbb{Z}_+, \tag{3.2}$$

where γ is a fixed integer, and that h_0 acts semisimply on V with integral eigenvalues. Then by Proposition 3.1, h_0 acts semisimply on any weak V-module. Combining the commutator formula with the condition (3.2) we get

$$[h_m, h_n] = m\gamma\delta_{m+n,0} \tag{3.3}$$

for $m, n \in \mathbb{Z}$.

From now on, we also freely use $h(n)$ for h_n. For any $\alpha \in \mathbb{Q}$, set

$$E^{\pm}(\alpha h, z) = \exp\left(\sum_{k=1}^{\infty} \frac{\alpha h(\pm k)}{k} z^{\mp k}\right). \tag{3.4}$$

Then from [LW] we have

$$E^{+}(\alpha h, z_1)E^{-}(\beta h, z_2) = \left(1 - \frac{z_2}{z_1}\right)^{-\gamma\alpha\beta} E^{-}(\beta h, z_2)E^{+}(\alpha h, z_1). \tag{3.5}$$

Define

$$\Delta(h, z) = z^{h(0)} \exp\left(\sum_{k=1}^{\infty} \frac{h(k)}{-k}(-z)^{-k}\right) = z^{h(0)}E^{+}(-h, -z) \in U(V)\{z\}. \tag{3.6}$$

PROPOSITION 3.2. *Let V be a regular vertex operator algebra and let $h \in V$ satisfying* (3.2). *Then $\Delta(h, z) \in G^0(V)$.*

To prove Proposition 3.2 we first prove the following lemma.

LEMMA 3.3. *Let $h \in V$ satisfying* (3.2). *Then we have*

$$E^{+}(h, z_1)Y(a, z_2)E^{+}(-h, z_1) = Y\left(z_1^{h(0)}\Delta(-h, z_1 - z_2)a, z_2\right) \quad \textit{for } a \in V. \tag{3.7}$$

Proof. For any $a \in V$, using the formula

$$[h(k), Y(a, z)] = \sum_{i=0}^{\infty} \binom{k}{i} z^{k-1} Y(h(i)a, z)$$

we obtain

$$
\begin{aligned}
&\left[\sum_{k=1}^{\infty} \frac{h(k)}{k} z_1^{-k}, Y(a, z_2)\right] \\
&\quad = \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \frac{1}{k} \binom{k}{i} z_1^{-k} z_2^{k-i} Y\big(h(i)a, z_2\big) \\
&\quad = \sum_{k=1}^{\infty} \frac{1}{k} z_1^{-k} z_2^{k} Y\big(h(0)a, z_2\big) + \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k} \binom{k}{i} z_1^{-k} z_2^{k-i} Y\big(h(i)a, z_2\big) \\
&\quad = -\log\left(1 - \frac{z_2}{z_1}\right) Y\big(h(0)a, z_2\big) \\
&\qquad + \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \frac{1}{k+i} \binom{k+i}{i} z_1^{-k-i} z_2^{k} Y\big(h(i)a, z_2\big) \\
&\quad = -\log\left(1 - \frac{z_2}{z_1}\right) Y\big(h(0)a, z_2\big) \\
&\qquad + \sum_{i=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{i} (-1)^k \binom{-i}{k} z_1^{-k-i} z_2^{k} Y\big(h(i)a, z_2\big) \\
&\quad = -\log\left(1 - \frac{z_2}{z_1}\right) Y\big(h(0)a, z_2\big) + \sum_{i=1}^{\infty} \frac{1}{i} (z_1 - z_2)^{-i} Y\big(h(i)a, z_2\big).
\end{aligned}
\tag{3.8}
$$

Then

$$
\begin{aligned}
&E^{+}(h, z_1) Y(a, z_2) E^{+}(-h, z_1) \\
&\quad = Y\left(\left(1 - \frac{z_2}{z_1}\right)^{-h(0)} E^{+}(h, z_1 - z_2) a, z_2\right) \\
&\quad = Y\big(z_1^{h(0)} \Delta(-h, z_1 - z_2) a, z_2\big). \quad \blacksquare
\end{aligned}
\tag{3.9}
$$

Proof of Proposition 3.2. Since $[h(0), Y(u, z)] = Y(h(0)u, z)$ for any $u \in V$, we have

$$
z^{h(0)} Y(u, z) z^{-h(0)} = Y(z^{h(0)} u, z) \qquad \text{for any } u \in V. \tag{3.10}
$$

Then it follows from the construction of $\Delta(h, z)$ and Lemma 3.3 that $\Delta(h, z)$ satisfies (2.14). Since

$$
[L(-1), h(0)] = 0, \qquad [L(-1), h(k)] = -kh(k-1) \text{ for } k \in \mathbb{Z},
$$

we obtain

$$[L(-1), \Delta(h,z)] = \sum_{k=1}^{\infty} h(k-1)(-z)^{-k}\Delta(h,z) = \frac{d}{dz}\Delta(h,z).$$

It is clear that $\Delta(h,z)$ satisfies the other conditions. Thus $\Delta(h,z) \in G^0(V)$. ∎

At the end of this section, we apply our results to some concrete examples. Let L be a positive-definite even lattice, let P be the dual lattice of L, and let V_L be the vertex operator algebra constructed by Borcherds [B] and Frenkel, Lepowsky, and Meurman [FLM2]. Then there is a 1-1 correspondence between the set of equivalence classes of irreducible modules for V_L and the set of cosets of P/L [B, FLM2, D]. More specifically, V_P is a V_L-module with the following decomposition into irreducible modules,

$$V_P = V_L \oplus V_{L+\beta_1} \oplus \cdots \oplus V_{L+\beta_{k-1}}, \tag{3.11}$$

where $k = |P/L|$.

PROPOSITION 3.4. *Let $\beta \in P$. Then as a V_L-module, $(V_L, Y(\Delta(\beta,z)\cdot, z))$ is isomorphic to the V_L-module $V_{L+\beta}$.*

Proof. For any $h' \in H = \mathbb{C} \otimes_{\mathbb{Z}} L$, we have

$$\Delta(\beta,z)h' = \Delta(\beta,z)h'(-1)\mathbf{1} = h' + z^{-1}\beta(h'). \tag{3.12}$$

Then $Y(\Delta(\beta,z)h', z) = Y(h',z) + z^{-1}\beta(h')$. Thus $\tilde{V}_L$ as a module for the Heisenberg algebra $\tilde{H} = \sum_{n \in \mathbb{Z}-\{0\}} t^n \otimes H \oplus \mathbb{C}$ is a completely reducible module which is isomorphic to V_L, and the set of eigenfunctions of $H(0)$ on $\tilde{V}_L$ is $\beta + L$. Then it follows from Theorem 3.1 of [D] that $(V_L, Y(\Delta(\beta,z)\cdot, z))$ is isomorphic to $V_{L+\beta}$. ∎

It follows from Proposition 3.4 that all irreducible V_L-modules can be obtained by using some $\Delta(\beta,z)$ and that $(V_L, Y(\Delta(\beta,z)\cdot, z))$ is isomorphic to $(V_L, Y(\cdot, z))$ if and only if $\beta \in L$. It is clear that $\Delta(\beta,z)$ is invertible so that each irreducible module is a simple current. It is also clear that $\Delta(\alpha,z)\Delta(\beta,z) = \Delta(\alpha+\beta,z)$ for $\alpha, \beta \in P$.

Let $\beta_i \in P$ $(i = 1,2)$. Then $Y(\Delta(\beta_2,z)\cdot, z)$ is a nonzero intertwining operator of type

$$\begin{pmatrix} (V_L, Y(\Delta(\beta_1+\beta_2,z)\cdot, z)) \\ (V_L, Y(\Delta(\beta_1,z)\cdot, z))(V_L, Y(\Delta(\beta_2,z)\cdot, z)) \end{pmatrix}.$$

Since each irreducible V_L-module is a simple current, all fusion rules are either zero or 1. This result on fusion rules has been obtained in Chapter 12 of [DL] using a different method. It is clear that Conjectures 2.8 and 2.13 hold for $V = V_L$.

Let $\mathbf{g}$ be a finite-dimensional simple Lie algebra with a fixed Cartan subalgebra H, let $\{\alpha_1, \dots, \alpha_n\}$ be a set of positive roots, and let $\{e_i, f_i, h_i \mid i = 1, \dots, n\}$ be the Chevalley generators. Let $\theta = \sum_{i=1}^{n} a_i \alpha_i$ be the highest positive root and let Ω be the dual Coxeter number of $\mathbf{g}$. Let $\langle \cdot, \cdot \rangle$ be the normalized Killing form on $\mathbf{g}$ such that $\langle \theta, \theta \rangle = 2$. Let λ_i $(i = 1, \dots, n)$ be the fundamental weights of $\mathbf{g}$ and let P_+ be the set of dominant integral weights of $\mathbf{g}$. A dominant integral weight λ is said to be *minimal* if it is minimal in P_+ (Exercise 13 in Section 13 of [Hum]). λ is said to be *cominimal* [FG] if $\lambda^\vee$ is minimal for the dual Lie algebra. From [K, Table Aff 1, Chap. 4], λ_i is cominimal if and only if $a_i = 1$. Let $\tilde{\mathbf{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathbf{g} \oplus \mathbb{C}c$ be the affine Lie algebra in [K]. For any complex number l and any weight λ of $\mathbf{g}$, let $L(l, \lambda)$ be the irreducible highest weight $\tilde{\mathbf{g}}$-module of level l with lowest weight λ. It has been well known (cf. [Li1, Theorem 4.3.1]; see also [FZ]) that $L(l, 0)$ has a natural vertex operator algebra structure if $l \neq -\Omega$.

PROPOSITION 3.5. *For any positive integer l, $L(l, l\lambda_i)$ is a simple current for $L(l, 0)$ if λ_i is cominimal.*

Proof. Choose $h \in H$ such that $\alpha_j(h) = \langle h, h_j \rangle = \delta_{i,j}$ for $1 \le j \le n$. Then we are going to show that $(V, Y(\Delta(h, z) \cdot, z))$ is isomorphic to $L(l\Lambda_i)$. Since $a_i = 1$, $\theta(h) = a_i = 1$. By definition we have

$$\Delta(h, z) h_j = h_j + l\delta_{i,j} z^{-1}, \qquad \Delta(h, z) e_i = z e_i, \qquad \Delta(h, z) f_i = z^{-1} f_i, \tag{3.13}$$

$$\Delta(h, z) e_j = e_j, \qquad \Delta(h, z) f_j = f_j, \qquad \Delta(h, z) f_\theta = z^{-1} f_\theta \text{ for } j \neq i. \tag{3.14}$$

In other words, the corresponding automorphism ψ of $U(\tilde{\mathbf{g}})$ or $U(L(l, 0))$ satisfies the conditions

$$\psi(h_i(n)) = h_i(n) + \delta_{n,0} l, \qquad \psi(e_i(n)) = e_i(n+1), \qquad \psi(f_i(n)) = f_i(n-1); \tag{3.15}$$

$$\psi(h_j(n)) = h_j(n), \qquad \psi(e_j(n)) = e_j(n), \qquad \psi(f_j(n)) = f_j(n) \text{ for } j \neq i, n \in \mathbb{Z}, \tag{3.16}$$

and

$$\psi(f_\theta(n)) = f_\theta(n-1) \qquad \text{for } n \in \mathbb{Z}. \tag{3.17}$$

Then the vacuum vector **1** in $(V, Y(\Delta(h, z)\cdot, z))$ is a highest weight vector of weight $l\lambda_i$. Thus $(V, Y(\Delta(h, z)\cdot, z))$ is isomorphic to $L(l, l\lambda_i)$ as a $\tilde{\mathfrak{g}}$-module. By Theorem 2.15, $L(l, l\lambda_i)$ is a simple current. ∎

Remark 3.6. Proposition 3.5 has been proved in [FG] by calculating the four point functions.

Remark 3.7. It has been proved in [F] that those are all simple currents except for E_8.

Remark 3.8. From Propositions 3.2, 2.1, and the proof of Proposition 3.5 we find that $L(l, l\lambda_i)$ is always a weak $L(l, 0)$-module for any complex number $l \neq -\Omega$.

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